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# **SOLUTION OF MAXWELL'S EQUATIONS IN A MAGNETO-IONIC MEDIUM WITH SOURCES**

R. MITTRA

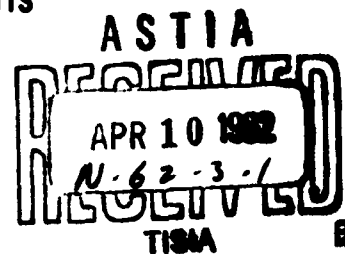
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Antenna Laboratory  
Electrical Engineering Research Laboratory  
Engineering Experiment Station  
University of Illinois  
Urbana, Illinois

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# SOLUTION OF MAXWELL'S EQUATIONS IN A MAGNETO-IONIC MEDIUM WITH SOURCES

## 1. Introduction

The knowledge of field solutions of Maxwell's Equations with source terms is of considerable importance for many applications. In this paper we have derived a complete solution of the field equations in a form which is suitable for the calculation of the fields of arbitrary distances from the source. For an infinitesimal dipole, the very near fields are found in a closed form involving trigonometric functions. The remainder of the solution is expressed in terms of a finite range integral, the integrand of which is finite everywhere. Hence, the solution is very suitable for numerical calculations. It is also pertinent to point out that the solutions derived hold for a very general nature of the elements of the  $\bar{\epsilon}$  tensor; for instance, the solutions are valid for complex elements and hence are useful for the treatment of a medium with finite losses. Two other previous contributions on this topic, e.g. by Bunkin<sup>1</sup> and Kogelnik<sup>2</sup> do not give a general solution such as the one derived in this paper.

## 2. Derivation of Matrix Equations in a Magneto-Ionic Medium with Sources

In this section we shall derive the matrix equations for the fields in a homogeneous medium with tensor dielectric properties, for the case of impressed electric current sources. The equations will subsequently be solved using the three dimensional Fourier transforms. Maxwell's Equations for the  $e^{j\omega t}$  time convention are:

$$\nabla \times \bar{E} = -j\omega \bar{H} \quad (1)$$

$$\nabla \times \bar{H} = j\omega \epsilon_0 \bar{\epsilon} \bar{E} + \bar{J} \quad (2)$$

$$\nabla \cdot \bar{H} = 0 \quad (3)$$

$$\nabla \cdot \bar{\epsilon} \bar{E} = \frac{\rho}{\epsilon_0} \quad (4)$$

where  $\bar{J}$  represents the impressed electric current source term. It will be assumed that the coordinate axes have been so oriented that  $\bar{\epsilon}$  has the form.

$$\bar{\epsilon} = \hat{x} \hat{x} \epsilon - j \hat{x} \hat{y} \epsilon' + j \hat{y} \hat{x} \epsilon' + \hat{y} \hat{y} \epsilon + \hat{z} \hat{z} \epsilon_z \quad (5)$$

where  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  are unit vectors in the cartesian system and  $\epsilon_0$  is the free space dielectric constant.

It is well known that in a megneto-ionic medium such a form for  $\bar{\epsilon}$  results when the z-direction is oriented along the DC magnetic field.

As a first step we shall point out why the conventional approach of introducing the vector potential  $\bar{A}$  through the relation

$$\bar{H} = \nabla \times \bar{A} \quad (6)$$

which satisfies (3), is not helpful in reducing the field equations to a simpler form. Using (6) in (1) we readily obtain the standard form

$$\bar{E} = -j\omega_\mu \bar{A} + \nabla V \quad (7)$$

and the substitution of (6) and (7) in (2) gives

$$\nabla \times \nabla \times \bar{A} = \nabla \nabla \cdot \bar{A} - \nabla^2 \bar{A} = \omega_\mu^2 \epsilon_0 \bar{\epsilon} \bar{A} + j\omega \epsilon_0 \bar{\epsilon} \nabla V + \bar{J} \quad (8)$$

When  $\bar{\epsilon}$  is a scalar ( $\epsilon$ ), (8) is readily simplified by introducing the relation

$$\nabla \nabla \cdot \bar{A} = j\omega \epsilon \epsilon_0 \nabla V$$

which reduces the equation for the vector potential to

$$\nabla^2 \bar{A} + \omega_\mu^2 \epsilon \epsilon_0 \bar{A} = -\bar{J}$$

However, the corresponding relation for the anisotropic case, viz.,

$$\nabla \nabla \cdot \bar{A} = j\omega \epsilon_0 \bar{\epsilon} \nabla V \quad (9)$$

is incompatible for a general  $\bar{\epsilon}$  as may be easily verified. Hence, as pointed out earlier, no particular advantage is gained by introducing the vector potential in the anisotropic case, and we shall find it convenient instead to treat the field equations directly. We now proceed to do this in the following.

Elimination of  $\bar{H}$  from (1) and (2) gives

$$\nabla \times \nabla \times \bar{E} = k_0^2 \bar{\epsilon} \bar{E} - j\omega_\mu \bar{J}, \quad k_0^2 = \omega_\mu^2 \epsilon_0 \quad (10)$$

Because of the particular form of  $\bar{\epsilon}$  we shall find it convenient to work with a different co-ordinate system in which the  $\bar{\epsilon}$  tensor diagonalizes. Such a system of co-ordinates is defined below. Let the unit vectors  $\hat{U}_1, \hat{U}_2, \hat{U}_3$  in the new system, hereafter called the U-system be defined by

$$\hat{U}_1 = \frac{1}{\sqrt{2}} (\hat{x} + j\hat{y}), \quad \hat{U}_2 = \frac{1}{\sqrt{2}} (\hat{x} - j\hat{y}), \quad \hat{U}_3 = \hat{z} \quad (11)$$

It may be readily verified that the following vector relations are satisfied by  $\hat{U}_1, \hat{U}_2$  and  $\hat{U}_3$ .

$$\begin{aligned} \hat{U}_1 \cdot \hat{U}_1 &= 0 & \hat{U}_2 \cdot \hat{U}_2 &= 0 \\ \hat{U}_1 \times \hat{U}_1 &= 0 & \hat{U}_2 \times \hat{U}_2 &= 0 \\ \hat{U}_1 \cdot \hat{U}_2 &= 1 & \hat{U}_1 \cdot \hat{U}_3 &= 0 & \hat{U}_2 \cdot \hat{U}_3 &= 0, \quad \hat{U}_3 \cdot \hat{U}_3 = 1 \\ \hat{U}_1 \times \hat{U}_2 &= -j \hat{U}_3 & \hat{U}_3 \times \hat{U}_1 &= -j \hat{U}_1 \end{aligned} \quad (12)$$

The above relations will be found useful later.

An arbitrary vector  $\bar{F}$  given by

$$\bar{F} = \hat{x} F_x + \hat{y} F_y + \hat{z} F_z \quad (13)$$

can be written in the U-system as

$$\bar{F} = \hat{U}_1 F_1 + \hat{U}_2 F_2 + \hat{U}_3 F_3 \quad (14)$$

where  $F_1 = \bar{F} \cdot \hat{U}_2$ ,  $F_2 = \bar{F} \cdot \hat{U}_1$  and  $F_3 = \bar{F} \cdot \hat{U}_3$

It may be easily verified that

$$F_1 = \frac{1}{\sqrt{2}} (F_x - j F_y), \quad F_2 = \frac{1}{\sqrt{2}} (F_x + j F_y), \quad F_3 = F_z \quad (15)$$

Now the representation for  $\bar{\epsilon}$  in the U-system is

$$\bar{\epsilon} = \hat{U}_1 \hat{U}_2 \epsilon_1 + \hat{U}_2 \hat{U}_1 \epsilon_2 + \hat{U}_3 \hat{U}_3 \epsilon_3 \quad (16)$$

where

$$\epsilon_1 = \epsilon + \epsilon' \quad \epsilon_2 = \epsilon - \epsilon' \quad \epsilon_3 = \epsilon_z$$

The expression for the operator  $\nabla$  is as follows

$$\nabla = \hat{U}_1 d_1 + \hat{U}_2 d_2 + U_3 d_3 \quad (17)$$

where

$$d_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} - j \frac{\partial}{\partial y} \right) \quad d_2 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) \quad d_3 = \frac{\partial}{\partial z}$$

After going through some algebra and the use of (16) and (17), the inhomogeneous field equation for  $\bar{E}$  given in (10) may be written in the matrix form as

$$\left\{ \begin{array}{c} -k_o^2 \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} + \begin{bmatrix} -(d_1 d_2 + d_3^2) & d_1^2 & d_1 d_3 \\ d_2^2 & -(d_1 d_2 + d_3^2) d_2 d_3 \\ d_3 d_2 & d_1 d_3 & -2d_1 d_2 \end{bmatrix} \end{array} \right\} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = j\omega\mu \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} \quad (18)$$

The above are the desired matrix equations for the fields. The solution for (18) will be obtained by taking the three-dimensional Fourier transform of the equations and by subsequent inversion of the transforms. This is discussed in the next section.

### 3. Transform method of Solution

Let the three dimensional Fourier transform  $\epsilon$  of  $E$  be defined by

$$\epsilon_{1,2,3} = \iiint_{-\infty}^{\infty} E_{1,2,3}(x,y,z) e^{+j(k_x x + k_y y + k_z z)} dx dy dz \quad (19)$$

and let the transforms  $J_{1,2,3}$  of  $J_{1,2,3}$  be similarly defined. Using the property of the transforms, the differential operators  $d_1, d_2, d_3$  may be written as

$$d_{1,2} = -\frac{j}{\sqrt{2}} (k_x \mp j k_y) \quad d_3 = -j k_z \quad (20)$$

For convenience, we shall, introduce the polar form through the definitions.

$$k_x = \Gamma \sin \psi \cos \alpha \quad k_y = \Gamma \sin \psi \sin \alpha \quad k_z = \Gamma \cos \psi \quad (21)$$

and rewrite (20) as

$$d_{1,2} = -j \frac{\Gamma}{\sqrt{2}} \sin \psi e^{\pm j\alpha} \quad d_3 = -j \Gamma \cos \psi \quad (22)$$

Taking the transform of the matrix equation (18) we may then derive the desired equation.

$$\left\{ \begin{array}{c} -k_o^2 \left[ \begin{array}{ccc} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{array} \right] - \Gamma^2 \left[ \begin{array}{c} -(\frac{\sin^2 \psi}{2} + \cos^2 \psi) \\ \frac{1}{2} \sin^2 \psi e^{2j\alpha} \\ \frac{1}{\sqrt{2}} \sin \psi \cos \psi e^{j\alpha} \end{array} \right] \end{array} \right\} \left[ \begin{array}{c} \frac{1}{2} \sin^2 \psi e^{-2j\alpha} \\ -(\frac{\sin^2 \psi}{2} + \cos^2 \psi) \\ \frac{1}{\sqrt{2}} \sin \psi \cos \psi e^{-j\alpha} \end{array} \right] \quad (23)$$

$$\left[ \begin{array}{c} \frac{1}{\sqrt{2}} \sin \psi \cos \psi e^{-j\alpha} \\ \frac{1}{\sqrt{2}} \sin \psi \cos \psi e^{j\alpha} \\ -\sin^2 \psi \end{array} \right] \left[ \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{array} \right] = -j\omega \mu \left[ \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right]$$

The determinant  $\Delta$  of the matrix inside the curly brackets of equation (23), obtained after working out some algebra is expressed as

$$\begin{aligned} \Delta &= -k_o^2 \Gamma^4 (\epsilon \sin^2 \psi + \epsilon_3 \cos^2 \psi) + k_o^4 \Gamma^2 \left\{ \epsilon_3 \epsilon (1 + \cos^2 \psi) + (\epsilon^2 - \epsilon'^2) \sin^2 \psi \right\} \\ &\quad - k_o^6 (\epsilon^2 - \epsilon'^2) \epsilon_3 \\ &= -k_o^2 (\epsilon \sin^2 \psi + \epsilon_3 \cos^2 \psi) \left\{ \Gamma^2 - n_1^2(\psi) \right\} \left\{ \Gamma^2 - n_2^2(\psi) \right\} \end{aligned} \quad (24)$$

where  $n_1^2, n_2^2$  are the roots of  $\Delta$ . The roots are given by

$$n_{1,2}^2 = k_o^2 \frac{B(\psi) \pm \sqrt{B^2(\psi) - 4A(\psi) C(\psi)}}{2A(\psi)} \quad (25)$$

where

$$A(\psi) = \epsilon (\sin^2 \psi + \epsilon_3 \cos^2 \psi)$$

$$B(\psi) = \left\{ \epsilon_3 \epsilon (1 + \cos^2 \psi) + (\epsilon^2 - \epsilon'^2) \sin^2 \psi \right\}$$

$$C(\psi) = (\epsilon^2 - \epsilon'^2) \epsilon_3$$

It may be verified that  $n_1$  and  $n_2$  are the indices of refraction for the extraordinary and ordinary waves, respectively, in the plane wave case. This is to be expected since the condition that a finite solution of (23) exist even when  $\Delta = 0$  is that  $\gamma_1 = \gamma_2 = \gamma_3 = 0$ , which obviously corresponds to the plane wave case. It should also be pointed out that the coefficient of  $\Gamma^6$  in the expansion for  $\Delta$  is identically zero and hence the third order determinant in  $\Gamma^2$  has only two roots. The coefficient of  $\Gamma^6$  is zero because of the property of the operator  $(\nabla \times \nabla \times)$  which makes the matrix corresponding to this operator a singular one.

It may be verified that the eigenvalue equation

$$|M_1 - \lambda M_2| = 0 \quad (26)$$

where  $M_1$  and  $M_2$  are  $(n \times n)$  matrices, has only  $(n-1)$  finite roots if the rank of  $M_2$  is  $(n-1)$ . Another observation about  $\Delta$  is that it is independent of  $\alpha$ .

The solution of the fields in the transformed domain is now a straightforward step involving the inversion of the matrix operator in (23). In the matrix form the solution is written as

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = -j\omega\mu \begin{bmatrix} a_{11}(\psi)/\Delta & a_{12}(\psi)/\Delta & a_{13}(\psi)/\Delta \\ a_{21}(\psi)/\Delta & a_{22}(\psi)/\Delta & a_{23}(\psi)/\Delta \\ a_{31}(\psi)/\Delta & a_{32}(\psi)/\Delta & a_{33}(\psi)/\Delta \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \quad (27)$$

where

$$a_{11} = \frac{\Gamma^4}{2} \sin^2 \psi - \Gamma^2 k_o^2 \left\{ \epsilon_2 \sin^2 \psi + \epsilon_3 \left( \frac{1}{2} \sin^2 \psi + \cos^2 \psi \right) \right\} + k_o^4 \epsilon_3 \epsilon_2 \quad (28)$$

$$a_{12} = e^{-2j\alpha} \frac{\Gamma^2}{2} \sin^2 \psi (\Gamma^2 - k_o^2 \epsilon_3) \quad (29)$$

$$a_{13} = e^{-j\alpha} \frac{\Gamma^2}{\sqrt{2}} \sin \psi \cos \psi (\Gamma^2 - k_o^2 \epsilon_2) \quad (30)$$

$$a_{21} = e^{2j\alpha} \frac{\Gamma^2}{2} \sin^2 \psi (\Gamma^2 - k_o^2 \epsilon_3) \quad (31)$$

$$a_{22} = \frac{\Gamma^4}{2} \sin^2 \psi - \Gamma^2 k_o^2 \left\{ \epsilon_1 \sin^2 \psi + \epsilon_3 \left( \frac{1}{2} \sin^2 \psi + \cos^2 \psi \right) \right\} + k_o^4 \epsilon_1 \epsilon_3 \quad (32)$$

$$a_{23} = e^{j\alpha} \frac{\Gamma^2}{\sqrt{2}} \sin \psi \cos \psi (\Gamma^2 - k_o^2 \epsilon_1) \quad (33)$$

$$a_{31} = e^{j\alpha} \frac{\Gamma^2}{\sqrt{2}} \sin \psi \cos \psi (\Gamma^2 - k_o^2 \epsilon_2) \quad (34)$$

$$a_{32} = \frac{e^{-j\alpha}}{\sqrt{2}} \Gamma^2 (\Gamma^2 - k_o^2 \epsilon_1) \sin \psi \cos \psi \quad (35)$$

$$a_{33} = \Gamma^4 \cos^2 \psi - k_o^2 \Gamma^2 (\epsilon_2 + \epsilon_1) \left( \frac{1}{2} \sin^2 \psi + \cos^2 \psi \right) + k_o^4 \epsilon_1 \epsilon_2 \quad (36)$$

and

$$\epsilon_1 = \epsilon + \epsilon', \quad \epsilon_2 = \epsilon - \epsilon', \quad \epsilon_3 = \epsilon_z.$$

The main problem of evaluating the inverse transforms still remains and we discuss this in the following.

#### 4. Evaluation of Inverse Transforms

We are now ready to get into the core of the problem, namely that of evaluating the inverse transforms of the fields. It should be pointed out that the asymptotic evaluation of the inverses, at distances far away from the sources, is a relatively simple task and has been discussed by Bunkin<sup>1</sup>. In this paper we shall attempt to evaluate the integrals for arbitrary distances and our results will be useful for far, near and intermediate field calculations. To the knowledge of the author a complete solution such as this has not been discussed elsewhere.

Let us assume that the current source is an infinitesimal one, i.e.

$$\begin{aligned}\bar{J} &= (\hat{x} C_x + \hat{y} C_y + \hat{z} C_z) \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) \\ &= (\hat{U}_1 C_1 + \hat{U}_2 C_2 + \hat{U}_3 C_3) \delta(x - x_0) \delta(y - y_0) \delta(z - z_0)\end{aligned}\quad (37)$$

where  $\delta$  is the Dirac  $\delta$  and  $C_1, C_2, C_3$  and hence  $C_1, C_2$  and  $C_3$  are constants, and may be readily obtained from the orientation and moment of the dipole. For an arbitrary distribution of sources in space, an integration over the distribution would yield the desired result. For an infinitesimal source at the origin the transform components are readily obtained by the use of the property of  $\delta$  and are given by

$$J_{1,2,3} = C_{1,2,3} \quad (38)$$

For simplicity, we shall carry out the steps for evaluating the inverse transforms for the case,  $C_1 = 1$  and  $C_2 = C_3 = 0$ . It is obvious that through the use of superposition we can readily derive the complete solution when the current coefficients  $C_2, C_3$  are non-zero, by following through similar routines.

From (27), (28) and (38) we have

$$E_\eta = -j\omega\mu \frac{a_{n1}(\Gamma, \psi, \alpha)}{\Delta(\Gamma, \psi)} \quad n = 1, 2, 3 \quad (39)$$

The inverse transform of a function  $\mathcal{F}(r, \psi, \alpha)$  is given by



$$F = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \varphi(k_x, k_y, k_z) e^{-j(k_x x + k_y y + k_z z)} dk_x dk_y dk_z \quad (40)$$

or in the polar form as

$$F = \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \varphi(\Gamma, \psi, \alpha) e^{-j\Gamma R (\sin \theta \sin \psi \cos(\alpha - \varphi) + \cos \theta \cos \psi)} \Gamma^2 \sin \psi d\alpha d\psi d\Gamma \quad (41)$$

$$\Gamma^2 \sin \psi d\alpha d\psi d\Gamma$$

where  $x = R \sin \theta \cos \varphi$ ,  $y = R \sin \theta \sin \varphi$  and  $z = R \cos \theta$ .

Hence the inverse of (39) may be written as

$$E_n = \frac{-j\omega_\mu}{(2\pi)^3} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{a_{n1}}{\Delta} (\Gamma, \psi, \alpha) e^{-j\Gamma R (\sin \theta \sin \psi \cos(\alpha - \varphi) + \cos \theta \cos \psi)} \Gamma^2 \sin \psi d\alpha d\psi d\Gamma \quad (42)$$

Let us consider the case  $n = 1$ . From (28) we see that  $a_{11}$  is independent of  $\alpha$  hence we can carry out the integration in (42) with respect to  $\alpha$  giving

$$E_1 = - \frac{j\omega_\mu}{(2\pi)^2} \int_0^{\infty} \int_0^{\pi} \Gamma^2 \frac{a_{11}(\Gamma, \psi)}{\Delta} e^{j\Gamma R \cos \theta \cos \psi} J_0(\Gamma R \sin \theta \sin \psi) \sin \psi d\psi d\Gamma \quad (43)$$

where we have made use of one of the well known<sup>3</sup> integral representations for the Bessel function.

We rewrite (43) in the form

$$E_1 = + \frac{j\omega_\mu}{k_0^2 (2\pi)^2} \nabla^2 I(R, \theta) \quad (44)$$

where

$$I(R, \theta) = k_o^2 \int_0^\infty \int_0^\pi \frac{a_{11}}{\Delta} e^{-j\Gamma R p(\theta, \psi)} J_0(\Gamma R q(\theta, \psi)) \sin \psi \, d\psi \, d\Gamma$$

$$p(\theta, \psi) = \cos \theta \cos \psi \quad (45)$$

$$q(\theta, \psi) = \sin \theta \sin \psi$$

and  $\Delta^2$  is the Laplacian operator. It was necessary to reorganize the integral representation so as to enable us to change the order of integration, which we intend to do shortly. It should be noted that since  $a_{11}$  and  $\Delta$  are both of fourth order in  $\Gamma$ ,  $a_{11}/\Delta \rightarrow \text{constant}$  for large  $\Gamma$  and the convergence of the integral representation for  $I(R, \theta)$  in (44) is assured for all  $R \neq 0$ . We anticipate the type of behavior  $I(R, \theta) \rightarrow 0(1/R)$  as  $R \rightarrow 0$ , so we set out to separate the singular part. When this is accomplished,  $I(R, \theta)$  may be written as

$$I(R, \theta) = I_s(R, \theta) + I_f(R, \theta) \quad (46)$$

where  $I_s(R, \theta)$  is  $O(1/R)$  for small  $R$  and  $I_f$  is finite for all  $R$  and  $\theta$ . The advantage of doing this is twofold. First, we shall find that it is possible to evaluate  $I_s$  (the singular part of the integral) exactly and hence to estimate the very near field behavior, which is dominated by the singular part. The second reason is that this procedure will make it convenient for us as we shall see shortly, to perform the integration in the expression for  $\Delta^2 I_f$ .  
Separation and Evaluation of the Singular Part  $I_s(R, \theta)$ :

Let us consider the ratio  $a_{11}/\Delta$ . From (24) and (28) we have, after replacing  $\epsilon_1$  by  $\epsilon + \epsilon'$  and  $\epsilon_2$  by  $\epsilon - \epsilon'$

$$k_o^2 \frac{a_{11}}{\Delta} = - \frac{(1/2)\Gamma^4 \sin^2 \psi - \Gamma^2 k_o^2 \left\{ (\epsilon - \epsilon') \sin^2 \psi + \epsilon_3 (1/2 \sin^2 \psi + \cos^2 \psi) \right\} + k_o^4 \epsilon_3 (\epsilon - \epsilon')}{(\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi) (\Gamma^2 - n_1^2) (\Gamma^2 - n_2^2)}$$

$$\begin{aligned}
& \Gamma^2 \left[ -\frac{\sin^2 \psi}{2} (n_1^2 + n_2^2) + k_0^2 (\epsilon - \epsilon') \sin^2 \psi \right. \\
& \quad \left. + k_0^2 \epsilon_3 \left( \frac{1}{2} \sin^2 \psi + \cos^2 \psi \right) \right] - k_0^4 \epsilon_3 (\epsilon - \epsilon') + \\
& \quad (47) \\
k_0^2 \frac{a_{11}}{\Delta} = & -\frac{1}{2} \frac{\sin^2 \psi}{(\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi)} + \frac{+n_1^2 n_2^2 \frac{\sin^2 \psi}{2}}{(\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi) (\Gamma^2 - n_1^2) (\Gamma^2 - n_2^2)}
\end{aligned}$$

Now let us define  $I_s$  as

$$I_s = -\frac{1}{2} \int_0^\infty \int_0^\pi \frac{\sin^2 \psi \cdot \sin \psi}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} e^{-j\Gamma R p(\theta, \psi)} J_0(\Gamma R q(\theta, \psi)) d\psi d\Gamma \quad (48)$$

then

$$\begin{aligned}
\nabla^2 I_s = & +\frac{1}{2} \int_0^\infty \int_0^\pi \frac{\Gamma^2 \sin^3 \psi}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} e^{-j\Gamma R p} J_0(\Gamma R q) d\psi d\Gamma \\
= & -\frac{1}{2} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \int_0^\infty \int_0^\pi \sin \psi \frac{e^{-j\Gamma R p}}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} J_0(\Gamma R q) d\psi d\Gamma \quad (49)
\end{aligned}$$

The integral in (49) permits an exact evaluation as shown in the appendix. Using (A-4) we then have

$$\nabla^2 I_s = -\frac{\Pi}{2} \cdot \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \frac{\epsilon^{-1/2}}{(\epsilon z^2 + \epsilon_3 \rho^2)^{1/2}} \quad (50)$$

It is easy to verify, say by solving the static problem, that  $\nabla^2 I_s$  has the correct order of singularity at the origin. We also note that the expression for  $I_s$  is independent of  $\epsilon'$ , the off-diagonal term of the  $\bar{\epsilon}$  tensor.

As a next step we go on to evaluate  $\nabla^2 I_f$  where  $I_f$ , the finite part of  $I$ , is given by  $I_f = I - I_s$ .

Evaluation of  $\nabla^2 I_f$ :

Although  $I_f$  itself is finite at the origin, we shall find that  $\nabla^2 I_f$  is not and it has a singular part of the type  $O(\frac{1}{R})$ . We shall now attempt to separate this singularity from the expression of  $\nabla^2 I_f$  by the same technique used in the previous section. We shall obtain an exact representation for the singular part and a reduced integral for the regular part in  $\nabla^2 I_f$ .

Using (45), (47) and (48) we have the representation for  $I_f$ .

$$I_f(R, \theta) = \int_0^\pi \int_0^\infty \frac{S(\Gamma, \psi) e^{-j\Gamma R p(\theta, \psi)}}{(\Gamma^2 - n_1^2)(\Gamma^2 - n_2^2)} J_0(\Gamma R q(\theta, \psi)) \sin \psi d\Gamma d\psi \quad (51)$$

where

$$S(\Gamma, \psi) = \frac{\Gamma^2 \left[ -\frac{\sin^2 \psi}{2} (n_1^2 + n_2^2) + k_o^2 (\epsilon - \epsilon') \sin^2 \psi + k_o^2 \epsilon_3 \left( \frac{1}{2} \sin^2 \psi + \cos^2 \psi \right) \right] - k_o^4 \epsilon_3 (\epsilon - \epsilon') + n_1^2 n_2^2 \frac{\sin^2 \psi}{2}}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} \quad (52)$$

Differentiating under the integral we obtain, remembering  $p = \cos \theta \cos \psi$ ,  $q = \sin \theta \sin \psi$

$$\nabla^2 I_f = - \int_0^\pi \int_0^\infty \frac{\Gamma^2 S(\Gamma, \psi)}{(\Gamma^2 - n_1^2)(\Gamma^2 - n_2^2)} e^{-j\Gamma R p} J_0(\Gamma R q) \sin \psi d\Gamma d\psi \quad (53)$$

Using the expression for  $S(\Gamma, \psi)$  from (52), break the integral in (53) into two parts as follows

$$\nabla^2 I_f = \int_0^\pi \int_0^\infty \frac{M(\psi) e^{-j\Gamma R p}}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} J_0(\Gamma R q) \sin \psi d\Gamma d\psi$$

$$- \int_0^\pi \int_0^\infty \frac{N(\Gamma, \psi)}{(\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi)(\Gamma^2 - n_1^2)(\Gamma^2 - n_2^2)} e^{-j\Gamma R p} J_0(\Gamma R q) \sin \psi d\Gamma d\psi \quad (54)$$

where

$$M(\psi) = [ - \sin^2 \psi (n_1^2 + n_2^2)/2 + k_o^2 (\epsilon - \epsilon') \sin^2 \psi + k_o^2 \epsilon_3 (\frac{1}{2} \sin^2 \psi + \cos^2 \psi) ] \quad (55)$$

$$N(\Gamma, \psi) = [ k_o^4 \epsilon_3 (\epsilon - \epsilon') - \frac{1}{2} n_1^2 n_2^2 \sin^2 \psi - (n_1^2 + n_2^2) M(\psi) ] \Gamma^2 + M(\psi) n_1^2 n_2^2 \quad (56)$$

The first integral which has the behavior of the type  $O(\frac{1}{R})$  at the origin permits an exact evaluation. The integration with respect to  $\Gamma$  in the second integral may be carried out using the techniques of contour integration, and an integral over a finite range may be derived. The resulting finite range integral is very suitable for numerical computations because the integrand is bounded in the entire range. The detailed calculation of the first integral and the reduction of the second integral are given in Appendices B and C, respectively. We shall merely present the final result which is

$$\begin{aligned} \nabla^2 I_f &= \frac{\pi k_o^2}{2} \frac{1}{R} \cdot Q(\theta) \\ &- 2\pi j \int_0^{\pi/2} \frac{N(n_1(\psi), \psi)}{A(\psi)} \frac{e^{-jn_1 R p} J_0(n_1 q R)}{n_1(n_1^2 - n_2^2)} \sin \psi d\psi \\ &+ 4 \int_{\pi/2-\theta}^{\pi/2} \int_0^{\cos^{-1}(p/q)} \frac{N(n_1(\psi), \psi)}{A(\psi)} \frac{\sin \{ n_1 R (p - q \cos \alpha) \}}{n_1(n_1^2 - n_2^2)} \sin \psi d\alpha d\psi \\ &+ P_{(n_2)}(R, \theta) \end{aligned} \quad (57)$$

$A(\psi) = \epsilon \sin^2 \psi + \epsilon_3 \cos^2 \psi$ ,  $Q(\theta)$  is given in (A-10) and  $P_{(n_2)}(R, \theta)$  is obtained by interchanging  $n_2$  and  $n_1$  everywhere in the two integrals appearing in (57). No further reduction of the integrals seems possible for a general choice of elements in the  $\bar{\epsilon}$  tensor, and numerical integration must be performed to evaluate the regular part of  $\nabla^2 I_f$ . Although we shall not work out the details, it is nevertheless of interest to point out that the two integrals give rise to an expression of the type

$$\frac{e^{-jkR}}{R} - \frac{1}{R}$$

when the medium under consideration is isotropic. It is also pertinent to mention here that if one is only interested in the far fields, these can be obtained by making the asymptotic approximations for large  $R$  in the integrals involved.

The desired final form for  $E_1$  is obtained by substituting (50) and (57) in (44). We shall therefore conclude our discussion of the determination of  $E_1$  here.

The other components of the electric field may be evaluated by following through a similar procedure.

##### 5. Comments and Discussions

In this paper we have obtained solutions of Maxwell's Equations with source terms, in a medium with linear dielectric properties. The very near field terms which have a singularity at the origin have been derived in an exact form and an integral representation for the regular part has been obtained. Since no restriction was put on the nature of the elements of the  $\bar{\epsilon}$  tensor, the solutions are valid for complex elements and hence they are useful for treating a medium with finite losses. Although the details are left out here, it is pertinent to point out that the integral representation of the regular part may be asymptotically evaluated for far field calculations. It is also of interest to mention here that, where  $\epsilon$  and  $\epsilon_3$  are pure real and have opposite signs, discontinuous types of solutions result because of the modification of the integrals involved. This aspect of the behavior of the solutions is under current investigation at the Antenna Laboratory by Mr. Keith Balmain and others.

## REFERENCES

1. Bunkin, "On Radiation in Anisotropic Media", J. Exptl. Theoret. Phys. (USSR) 32, pp. 338-346, Feb., 1957.
2. Kogelnik, "On Electromagnetic Radiation in Magneto-ionic Media", J. of Res., NBS Vol. 64D, No. 5, Sept.-Oct., 1960, pp. 515-522.
3. Magnus and Oberhettinger, "Functions of Mathematical Physics", Chelsea Publishing Co., 1954, p. 26.
4. Magnus and Oberhettinger, op. cit., p. 33.

## APPENDIX A

Evaluation of the integral appearing in (49):

The integral to be evaluated, say  $L(R, \theta)$  is

$$L(R, \theta) = \int_0^{\infty} \int_0^{\pi} \frac{e^{-j\Gamma R \cos \theta \cos \psi}}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} J_0(\Gamma R \sin \theta \sin \psi) \sin \psi \, d\psi \, d\Gamma \quad (A-1)$$

$$= \int_0^{\infty} \int_0^{\pi} \frac{e^{-j\Gamma R \cos \theta \cos \psi}}{\epsilon_3 \Gamma^2 \cos^2 \psi + \epsilon \Gamma^2 \sin^2 \psi} J_0(\Gamma R \sin \theta \sin \psi) \Gamma^2 \sin \psi \, d\Gamma \, d\psi$$

Introducing a change in variables through the relations.

$$\gamma = \Gamma \cos \psi, \quad u = \Gamma \sin \psi$$

$$z = R \cos \theta, \quad \rho = R \sin \theta$$

we derive

$$L(R, \theta) = \int_0^{\infty} \int_{-\infty}^{\infty} \frac{e^{-j\gamma z} J_0(u\rho)}{\epsilon_3 \gamma^2 + \epsilon u^2} \, u \, d\gamma \, du \quad (A-2)$$

Performing the integration with respect to  $\gamma$  we obtain through the use of contour integration methods

$$L(R, \theta) = \pi \int_0^{\infty} \frac{e^{-(\epsilon/\epsilon_3)^{1/2} uz} J_0(u\rho)}{(\epsilon \epsilon_3)^{1/2}} \, du \quad (A-3)$$

(A-3) is a standard integral and is discussed in books on Bessel functions<sup>4</sup>. The final form for  $L(R, \theta)$  reads



$$L(R, \theta) = \frac{\pi}{(\epsilon \epsilon_3)^{1/2}} \frac{1}{\left\{ (\epsilon/\epsilon_3) z^2 + \rho^2 \right\}^{1/2}}$$

(A-4)

$$= \frac{1}{\epsilon^{1/2}} \frac{\pi}{R(\epsilon \cos^2 \theta + \epsilon_3 \sin^2 \theta)^{1/2}},$$

Real  $(\epsilon/\epsilon_3)^{1/2} > 0$

## APPENDIX B

Evaluation of the singular part of  $\nabla^2 I_f$ :

As pointed out in connection with (54), the singular part of  $\nabla^2 I_f$  can be evaluated exactly. Let the singular part be represented by  $K(R, \theta)$  where

$$K(R, \theta) = \int_0^\pi \int_0^\infty \frac{M(\psi) e^{-j\Gamma R p}}{\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi} J_0(\Gamma R q) \sin \psi d\Gamma d\psi \quad (B-1)$$

and  $p = \cos \theta \cos \psi$ ,  $q = \sin \theta \sin \psi$ .

As a first step toward evaluating the integral, substitute the expressions for  $n_1$  and  $n_2$  from (25) into (55) and rewrite  $M(\psi)$  as

$$M(\psi) = \frac{k_o^2}{2} (\epsilon_3 \cos^2 \psi + \epsilon \sin^2 \psi)^{-1} \times \left\{ \sin^4 \psi (\epsilon - \epsilon')^2 + \sin^2 \psi \cos^2 \psi \epsilon_3 (\epsilon_3 + 2\epsilon - 2\epsilon') + 2\epsilon_3^2 \cos^4 \psi \right\} \quad (B-2)$$

Now multiply the denominator and the numerator of the integrand in (A-5) by  $\Gamma^6$  and introduce the transformations

$$\gamma = \Gamma \cos \psi \quad u = \Gamma \sin \psi$$

and

$$z = R \cos \theta \quad \rho = R \sin \theta$$

obtaining

$$2K(R, \theta) = k_o^2 \int_0^\infty \int_{-\infty}^\infty \frac{\left\{ u^4 (\epsilon - \epsilon')^2 + u^2 \gamma^2 \epsilon_3 (\epsilon_3 + 2\epsilon - 2\epsilon') + 2\epsilon_3^2 \gamma^4 \right\}}{(\epsilon_3 \gamma^2 + \epsilon u^2)^2 (\gamma^2 + u^2)} \times e^{-j\gamma z} J_0(\rho) u d\gamma du \quad (B-3)$$

Evaluate the integral with respect to  $\gamma$  using the method of contour integration and obtain

$$\begin{aligned}
2k_0^{-2} K(R, \theta) = & \pi \frac{\epsilon^2 + \epsilon'^2 + \epsilon_3^2 - 2(\epsilon\epsilon' + \epsilon_3\epsilon - \epsilon_3\epsilon')}{(\epsilon - \epsilon_3)^2} \int_0^\infty e^{-uz} J_0(u\rho) du \\
& + \pi \frac{(\epsilon^3 - 2\epsilon^2\epsilon_3 + 4\epsilon^2\epsilon' - 4\epsilon\epsilon_3\epsilon + \epsilon\epsilon_3^2 + \epsilon_3\epsilon'^2 - 3\epsilon\epsilon'^2)}{2\left(\frac{\epsilon}{\epsilon_3}\right)^{1/2} \epsilon(\epsilon - \epsilon_3)^2} \\
& \times \int_0^\infty e^{-u(\epsilon/\epsilon_3)^{1/2}z} J_0(u\rho) du \\
& - \left(\frac{\pi z}{2}\right) \frac{\epsilon^2 + \epsilon'^2 - \epsilon\epsilon_3}{\epsilon(\epsilon - \epsilon_3)} \int_0^\infty e^{-u(\epsilon/\epsilon_3)^{1/2}z} J_0(u\rho) u du
\end{aligned} \tag{B-4}$$

The first term in (B-4) is the contribution of the pole at  $\gamma = -ju$  whereas the second and third terms are due to the residue of the double pole at  $\gamma = -j(\epsilon/\epsilon_3)^{1/2}u$ .

Now rewrite the third term as

$$\int_0^\infty e^{-zu(\epsilon/\epsilon_3)^{1/2}} J_0(u\rho) u du = - \left(\frac{\epsilon_3}{\epsilon}\right)^{1/2} \frac{\delta}{\delta z} \int_0^\infty e^{-zu(\epsilon/\epsilon_3)^{1/2}} J_0(u\rho) du \tag{B-5}$$

and evaluate the integrals involving the Bessel functions in the same manner as in Appendix A, obtaining after some simplification

$$\begin{aligned}
2\pi^{-1} k_0^{-2} K(R, \theta) = & \frac{1}{R} \left\{ \frac{\epsilon^2 + \epsilon'^2 + \epsilon_3^2 - 2(\epsilon\epsilon' + \epsilon_3\epsilon - \epsilon_3\epsilon')}{(\epsilon - \epsilon_3)^2} \right. \\
& + \frac{1}{2} \frac{\epsilon_3}{\epsilon^{3/2}} \cdot \frac{\epsilon^2 - 2\epsilon\epsilon_3 + \epsilon\epsilon_3^2 + 4\epsilon^2\epsilon' + \epsilon_3\epsilon'^2 - 3\epsilon\epsilon'^2 - 4\epsilon\epsilon\epsilon'}{(\epsilon - \epsilon_3)^2} \left. \right\}
\end{aligned} \tag{B-6}$$

$$\begin{aligned}
& \cdot \frac{1}{(\epsilon \cos^2 \theta + \epsilon_3 \sin^2 \theta)^{1/2}} \\
& - \frac{1}{2} \frac{\epsilon_3}{\epsilon^{1/2}} \frac{\epsilon^2 + \epsilon^2 - \epsilon \epsilon_3}{(\epsilon - \epsilon_3)} \frac{\cos^2 \theta}{(\epsilon \cos^2 \theta + \epsilon_3 \sin^2 \theta)^{3/2}} \} \quad (B-6)
\end{aligned}$$

$$= \frac{1}{R} Q(\theta)$$

where  $Q(\theta)$  is the expression inside the curly brackets.

### REDUCTION OF REGULAR PART OF $\nabla^2 I_f$

We shall discuss the reduction of the integral say  $P(R, \theta)$ , which is the regular part of  $\nabla^2 I_f$  given in (54). Let

$$P(R, \theta) = \int_0^\pi \int_0^\infty \frac{N(\Gamma, \psi) \sin \psi e^{-j\Gamma R p}}{A(\psi) (\Gamma^2 - n_1^2) (\Gamma^2 - n_2^2)} J_0(\Gamma R q) d\Gamma d\psi \quad (C-1)$$

$$A(\psi) = \epsilon \sin^2 \psi + \epsilon_3 \cos^2 \psi$$

We note that the highest power of  $\Gamma$  in  $N(\Gamma, \psi)$  is 2, hence the integral  $P(R, \theta)$  is finite even when  $R \rightarrow 0$ . To evaluate the integral, we recast it in the following form after a couple of simple changes of variables.

$$P(R, \theta) = 2 \int_0^{\pi/2} \int_{-\infty}^\infty \frac{N(\Gamma, \psi) \sin \psi e^{-j\Gamma R p}}{A(\psi) (\Gamma^2 - n_1^2) (\Gamma^2 - n_2^2)} J_0(\Gamma R q) d\Gamma d\psi \quad (C-2)$$

The advantage of the form of (C-2) over (C-1) is that (C-1) is suitable for integration with respect to  $\Gamma$  using the technique of contour integration because of infinite limits. Before this integration is performed, however, it is convenient to replace the Bessel function by its integral representation and arrive at the form

$$P(R, \theta) = \frac{2}{\pi} \int_0^{\pi/2} \int_{-\infty}^\infty \int_0^{\pi/2} \frac{N(\Gamma, \psi) \sin \psi}{A(\psi)} \frac{e^{-j\Gamma R (p + q \cos \alpha)}}{(\Gamma^2 - n_1^2) (\Gamma^2 - n_2^2)} d\alpha d\Gamma d\psi \\ + \frac{2}{\pi} \int_0^{\pi/2} \int_{-\infty}^\infty \int_0^{\pi/2} \frac{N(\Gamma, \psi) \sin \psi}{A(\psi)} \frac{e^{-j\Gamma R (p - q \cos \alpha)}}{(\Gamma^2 - n_1^2) (\Gamma^2 - n_2^2)} d\alpha d\Gamma d\psi \quad (C-3)$$

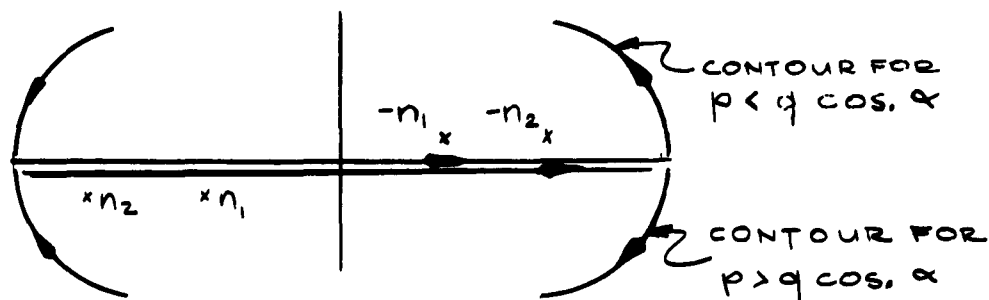


Figure 1. Location of poles and choice of contours for the second integral in (C-3)

Now interchange the order of integration so that the integration is first performed with respect to  $\Gamma$ . Following the usual procedure, extend the integral to a infinite semicircular contour over a complex  $\Gamma$  plane, closing the contour in the upper half plane when the coefficient of  $\Gamma$  in the exponent is positive and in the lower half plane when otherwise. Assuming  $n_1$  and  $n_2$  have finite negative imaginary parts, however small, we see that only the poles at  $\Gamma = n_1$  and  $n_2$  are enclosed in the lower half plane and their negative counterparts in the upper half plane. Note that  $p + q \cos \alpha$  is always positive for  $0 < \psi < \pi/2$  and  $0 < \alpha < \pi/2$  hence the first integral in (C-3) is always evaluated by closing the contour in the lower half plane. The evaluation of the integrals gives

$$P(R, \theta) = P_{(n_1)}(R, \theta) + P_{(n_2)}(R, \theta)$$

where  $P_{(n_1)}$  and  $P_{(n_2)}$  are contributions due to the poles  $n_1$  and  $n_2$  respectively.

The expression for  $P_{(n_1)}$  is

$$P_{(n_1)}(R, \theta) = -2j \int_0^{\pi/2} \int_0^{\pi/2} \frac{N(n_1(\psi), \psi) \sin \psi}{A(\psi)} \frac{e^{-jn_1 R(p + q \cos \alpha)}}{n_1(n_1^2 - n_2^2)} d\alpha d\psi \quad (C-4)$$

$$-2j \int_0^{\pi/2} \int_{\cos^{-1}(p/q)}^{\pi/2} \sin \psi \frac{N(n_1(\psi), \psi) \sin \psi}{A(\psi)} \frac{e^{-jn_1 R(p - q \cos \alpha)}}{n_1(n_1^2 - n_2^2)} d\alpha d\psi \quad (C-4)$$

$$- 2j \int_{\pi/2-\theta}^{\pi/2} \int_0^{\cos^{-1}(p/q)} \frac{N(n_1(\psi), \psi) \sin \psi}{A(\psi)} \frac{e^{jn_1 R(p - q \cos \alpha)}}{n_1(n_1^2 - n_2^2)} d\alpha d\psi$$

and  $P_{(n_2)}(R, \theta) = P_{(n_1)}(R, \theta)$  with  $n_2$  and  $n_1$  interchanged everywhere. The integrals in (C-4) may be further reduced. To this end rearrange (C-4) and derive

$$P_{(n_1)}(R, \theta) = -4j \int_0^{\pi/2} \int_0^{\pi/2} \frac{N(n_1(\psi), \psi)}{A(\psi)} \frac{e^{-jn_1 R p \cos(n_1 q \cos \alpha)}}{n_1(n_1^2 - n_2^2)} \sin \psi d\alpha d\psi \quad (C-5)$$

$$+ 4 \int_{\pi/2-\theta}^{\pi/2} \int_0^{\cos^{-1}(p/q)} \frac{N(n_1(\psi), \psi)}{A(\psi)} \frac{\sin \{n_1 R(p - q \cos \alpha)\}}{n_1(n_1^2 - n_2^2)} \sin \psi d\alpha d\psi$$

Note that in writing (C-5) we have made use of the fact that

$$\int_0^{\pi/2-\theta} \int_0^{\cos^{-1} p/q} F(\psi, \theta, \alpha) d\alpha d\psi = 0$$

since

$$p/q = \frac{\cos \theta \cos \psi}{\sin \theta \sin \psi} > 1, \quad \text{for } \psi < \pi/2 - \theta$$

and the integral with respect to  $\alpha$  is over a real range only.

The first integral with respect to  $\alpha$  may be reduced to a Bessel function yielding the final representation for  $P_{(n_1)}$  as given below

$$P_{(n_1)}(R, \theta) = -2\pi j \int_0^{\pi/2} \frac{N(n_1(\psi), \psi)}{A(\psi)} \frac{e^{-jn_1 R p} J_0(n_1 q R)}{n_1(n_1^2 - n_2^2)} \sin \psi \, d\psi$$

(C-6)

$$+ 4 \int_{\pi/2-\theta}^{\pi/2} \int_0^{\cos^{-1}(p/q)} \frac{N(n_1(\psi), \psi)}{A(\psi)} \frac{\sin \{ n_1 R (p-q \cos \alpha) \}}{n_1(n_1^2 - n_2^2)} \sin \psi \, d\alpha \, d\psi$$



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